

# Efficient Mechanisms for Level- $k$ Bilateral Trading

Vincent P. Crawford

University of Oxford, All Souls College,  
and University of California, San Diego



UC San Diego



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## Introduction

The paper revisits Myerson and Satterthwaite's 1983 *JET* ("MS") classic analysis of the design of incentive-efficient mechanisms for bilateral trading with independent private values, inspired by Chatterjee and Samuelson's 1983 *OR* ("CS") positive analysis.



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To focus on nonequilibrium thinking, I maintain standard rationality assumptions regarding decisions and probabilistic judgment.

## Motivation

- Mechanism design often creates novel games, weakening the learning justification for equilibrium; yet the design may need to work well the first time.
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- Even if learning is possible, design may create games complex enough that convergence to equilibrium is behaviorally unlikely.

We usually assume equilibrium anyway, perhaps because:

- We doubt we can identify a credible basis for analysis among the enormous number of possible nonequilibrium models.
- We doubt that any nonequilibrium model could systematically out-predict a rational-expectations notion such as equilibrium.

## But...

- There is now a large body of experimental research that studies strategic thinking by eliciting subjects' initial responses to games (surveyed in Crawford, Costa-Gomes, and Iriberri 2013 *JEL*).
- The evidence suggests that people's thinking in novel or complex games does not follow the fixed-point or indefinitely iterated dominance reasoning that equilibrium often requires.



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- To the extent that people do not follow equilibrium logic, they must find another way to think about the game.
- Much evidence points to a class of nonequilibrium level- $k$  or "cognitive hierarchy" models of strategic thinking.

## Level- $k$ models

In a level- $k$  model people follow rules of thumb that:

- Anchor their beliefs in a naïve model of others' response to the game, called  $L0$ , often uniform random over feasible decisions; and
- Adjust their beliefs via a small number ( $k$ ) of iterated best responses, so  $L1$  best responds to  $L0$ ,  $L2$  to  $L1$ , and so on.

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Estimates vary with the setting and population, but normally the estimated frequency of  $L0$  is small or zero and the distribution of levels is concentrated on  $L1$ ,  $L2$ , and  $L3$ .

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- Thus  $L_k$  mimics equilibrium decisions in  $k$ -dominance-solvable games, but may deviate systematically in more complex games.
- A level- $k$  model (with zero weight on  $L_0$ ) can be viewed as a heterogeneity-tolerant refinement of  $k$ -rationalizability.
- But unlike  $k$ -rationalizability, a level- $k$  model makes precise predictions, given the population level frequencies: not only that deviations from equilibrium will sometimes occur, but also which settings evoke them and which forms they are likely to take.

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- A level- $k$  analysis can identify settings where conclusions based on equilibrium are robust to likely deviations from equilibrium.
- A level- $k$  analysis could identify settings in which mechanisms that yield superior outcomes in equilibrium are worse in practice than others whose performance is less sensitive to deviations: an evidence-disciplined approach to robustness.

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- A level- $k$  analysis could identify settings in which mechanisms that yield superior outcomes in equilibrium are worse in practice than others whose performance is less sensitive to deviations: an evidence-disciplined approach to robustness.
- A level- $k$  analysis might reduce optimal mechanisms' sensitivity to distributional and other details that real mechanisms seldom depend on, as advocated by Robert Wilson (1987) and others.

## Antecedents

- Crawford and Iriberry's (2007 *ECMA*) level- $k$  analysis of bidding behavior in sealed-bid independent-private-value and common-value auctions, which builds on Milgrom and Weber's (1982 *ECMA*) equilibrium analysis.
- Crawford, Kugler, Neeman, and Pautner's (2009 *JEEA*; "CKNP") level- $k$  analysis of optimal independent-private-value auctions, which builds on Myerson's (1981 *MathOR*) equilibrium analysis.
- Saran's (2011 *GEB*) analysis of MS's design problem with a known population frequency of truthful traders.
- Kneeland's (2013) analysis of level- $k$  implementation, with illustrations including bilateral trading.
- Gorelkina's (2015) level- $k$  analysis of the expected externality mechanism.
- de Clippel, Saran, and Serrano's (2015) analysis of design with bounded depth of reasoning, and with small errors.

## Outline

- CS's equilibrium analysis of bilateral trading via double auction
- MS's analysis of equilibrium-incentive-efficient mechanisms
- A level- $k$  model for direct games with asymmetric information
- Level- $k$  analysis of the double auction
  - $L1$ s' optimism, aggressiveness, and incentive-*inefficiency*
  - $L2$ s' pessimism, *unaggressiveness*, incentive-*superefficiency*
- Design requiring level- $k$ -incentive-compatibility (not wlog here)
  - Levels known or observable
  - "Level- $k$  menu effects" and failures of the revelation principle
  - Levels unknown and unobservable, and heterogeneous
- Design relaxing level- $k$ -incentive-compatibility
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## **CS's equilibrium analysis of bilateral trading via double auction**

CS's model has a potential seller and buyer of an indivisible object, in exchange for money.

Their von Neumann-Morgenstern utility functions are quasilinear in money: risk-neutral, with money values for the object.



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Denote the buyer's value  $V$  and the seller's value  $C$  (for "cost").

$V$  and  $C$  are independent, with positive densities  $f(V)$  and  $g(C)$  on their supports and distribution functions  $F(V)$  and  $G(C)$ .

CS and MS allowed the densities to have any bounded overlapping supports, but without important loss of generality I take the supports to be identical and normalize them to  $[0, 1]$ .

In the double auction:

- If the buyer's money bid  $b \geq$  the seller's money ask  $a$ , the seller exchanges the object for a given weighted average of  $b$  and  $a$ .
- CS allowed any weights between 0 and 1, but I take the weights to be equal, so the buyer acquires the object at price  $(a + b)/2$ , the seller's utility is  $(a + b)/2$ , and the buyer's is  $V - (a + b)/2$ .
- If  $b < a$ , the seller retains the object, no money changes hands, the seller's utility is  $C$ , and the buyer's utility is 0.
- I ignore the possibility that  $a = b$ , which will have 0 probability.

The double auction has many Bayesian equilibria.

When  $f(V)$  and  $g(C)$  are uniformly distributed, CS identify a linear equilibrium, which also plays a central role in MS's analysis.

Denote the buyer's bidding strategy  $b(V)$  and the seller's asking strategy  $a(C)$ , with \* subscripts for the equilibrium strategies.

In the linear equilibrium, with value densities supported on  $[0, 1]$ ,

$$b_*(V) = 2V/3 + 1/12$$

unless  $V < 1/4$ , when  $b_*(V)$  can be anything that precludes trade;

and

$$a_*(C) = 2C/3 + 1/4$$

unless  $C > 3/4$ , when  $a_*(C)$  can be anything that precludes trade.

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Trade occurs if and only if  $2V/3 + 1/12 \geq 2C/3 + 1/4$ , or  $V \geq C + 1/4$ :  
with positive probability the outcome is ex post inefficient.

The ex ante probability of trade is  $9/32 \approx 28\%$  and the expected total surplus is  $9/64 \approx 0.14$ , less than the maximum individually rational probability of trade 50% and expected surplus  $1/6 \approx 0.17$ .

## **MS's analysis of equilibrium-incentive-efficient mechanisms**

MS characterized incentive-efficient mechanisms in CS's trading environment, requiring interim individual rationality.

Like CS, MS allowed general, independent value distributions with strictly positive densities on ranges that overlap for the buyer and seller; but I will continue to take both value supports to be  $[0, 1]$ .

MS assumed that traders will play any desired Bayesian equilibrium in the game created by the chosen mechanism.

A *direct* (or *direct-revelation*) mechanism is one in which players' decisions are conformable to estimates of their values, with the outcome a function of the reported values.

When traders are risk-neutral in money, denoting their value reports  $v$  and  $c$  (distinct from true values  $V$  and  $C$ ), the payoff-relevant aspects of an outcome are determined by two functions:

- $p(v, c)$ , the probability that the object is transferred, and
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A direct mechanism with outcome functions  $p(\cdot, \cdot)$ ,  $x(\cdot, \cdot)$  is *incentive-compatible* iff it makes truthful reporting an equilibrium; and is (*interim*) *individually rational* iff it yields buyer and seller expected utility  $\geq 0$  for every possible realization of their values.



The revelation principle shows that if traders can be counted on to play any desired equilibrium in the game created by the designer's chosen mechanism, there is no loss of generality in restricting attention to incentive-compatible direct mechanisms:

“We can, without any loss of generality, restrict our attention to incentive-compatible direct mechanisms. This is because, for any Bayesian equilibrium of any bargaining game, there is an equivalent incentive-compatible direct mechanism that always yields the same outcomes (when the individuals play the honest equilibrium)....[w]e can construct [such a] mechanism by first asking the buyer and seller each to confidentially report his valuation, then computing what each would have done in the given equilibrium strategies with these valuations, and then implementing the outcome (transfer of money and object) as in the given game for this computed behavior. If either individual had any incentive to lie to us in this direct mechanism, then he would have had an incentive to lie to himself in the original game, which is a contradiction of the premise that he was in equilibrium in the original game.” (MS, pp. 267-268)

MS's Theorem 1 uses the conditions for incentive-compatibility and individual rationality to derive an "incentive budget constraint" (my term, not theirs), subject to which, for traders with quasilinear utility functions, incentive-efficient outcome functions  $p(\cdot, \cdot)$  and  $x(\cdot, \cdot)$  must maximize the sum of traders' ex ante expected utilities.

MS's Theorem 2 uses Theorem 1's conditions to characterize the outcome functions associated with incentive-efficient mechanisms.

MS's Corollary 1 shows that no incentive-compatible, individually rational mechanism can assure ex post Pareto-efficiency with probability one.

(The level- $k$  counterparts of MS's results, explained below, will allow a more detailed exposition of MS's analysis.)

In CS's example with uniform value densities, MS's Theorem 2 yields a closed-form solution for the incentive-compatible, incentive-efficient mechanism, which transfers the object whenever the reported values satisfy  $v \geq c + \frac{1}{4}$ , at price  $(v + c + \frac{1}{2})/3$ .

The linear equilibrium of the double auction with uniform densities transfers the object whenever the *true* values satisfy  $V \geq C + \frac{1}{4}$ .

Thus, even though the double auction is not incentive-compatible, linear equilibrium bidding strategies shade to mimic the outcome of truthful reporting in MS's incentive-efficient mechanism.

(Satterthwaite and Williams 1989 *JET* showed, however, that for generic densities CS's double auction does *not* yield incentive-efficient outcomes; so MS's result for this example is coincidental.)

## **A level- $k$ model for direct games with asymmetric information**

- Recall that in a level- $k$  model people anchor their beliefs in a naïve model of others' responses,  $L0$ , and adjust their beliefs via iterated best responses:  $L1$  best responds to  $L0$ , and so on.
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- In complete-information games  $L0$  is usually assumed to make decisions uniformly distributed over the feasible decisions.
- Following Camerer, Ho, and Chong (2004 *QJE*), Crawford and Iriberri (2007 *ECMA*), and CKNP, I take  $L0$ 's decisions to be uniform over the feasible decisions, *independent of own value*.
- Specifically, in a direct mechanism for the bilateral trading setting, I take  $L0$ 's decisions to be uniform over the range  $[0, 1]$ . (If I allowed bounded overlapping supports as CS and MS did, my assumption corresponds to assuming that  $L0$  is uniform on the overlap, which traders have enough information to identify.)
- This  $L0$  yields a hierarchy of rules via iterated best responses.

- One can imagine more refined specifications, e.g. with an  $L0$  buyer's bid (seller's ask) uniform below (above) its value instead of over the entire range, thus eliminating dominated strategies.
- But  $L0$  is not an actual player: It is a player's naïve model of other players—others whose values he does not observe.
- It is logically possible that  $Lk$  players initially reason contingent on others' possible values, but behaviorally far-fetched.
- A level- $k$  model with  $L0$  uniform over the feasible decisions and independent of own value captures people's aversion to fixed-point and complex contingent reasoning in a tractable way.

This extended level- $k$  model has a long history.

Milgrom and Stokey's (1982 *JET*) "No-Trade Theorem" shows that if traders in an asset market start out in market equilibrium—Pareto-efficient, given their information—then giving them new information, fundamentals unchanged, cannot lead to new trades. For, any such trades would make it common knowledge that both had benefited, contradicting the efficiency of the initial equilibrium.

This result has been called the Groucho Marx Theorem:

“I sent the club a wire stating, ‘Please accept my resignation. I don’t want to belong to any club that will accept people like me as a member’.”

—Groucho Marx, Telegram to the Beverly Hills Friars’ Club

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In speculating on why zero-sum trades occur despite the theorem, Milgrom and Stokey contrast Groucho's equilibrium inference with their rules *Naïve Behavior*, which sticks with its prior but behaves rationally otherwise, as *L1* does; and *First-Order Sophistication*, which best responds to *Naïve Behavior*, just as *L2* does.



See also:

“Son...One of these days in your travels, a guy is going to show you a brand-new deck of cards on which the seal is not yet broken. Then this guy is going to offer to bet you that he can make the jack of spades jump out of this brand-new deck of cards and squirt cider in your ear. But, son, do not accept this bet, because as sure as you stand there, you're going to wind up with an ear full of cider.”

—Obadiah (“The Sky”) Masterson, quoting his father in Damon Runyon (*Guys and Dolls: The Stories of Damon Runyon*, 1932)

Dad is worried that young Sky is an  $L1$ : rational except for sticking with his prior in the face of an offer that is “too good to be true”.

There is clear evidence that this extended level- $k$  model gives a realistic account of the main patterns of people's strategic thinking and “informational naiveté”, their failure to attend to how others' incentives depend on their private information.

- Camerer et al. (2004 *QJE*) suggested that a cognitive hierarchy analogue of this model could explain zero-sum betting.
- Brocas, Carillo, Camerer, and Wang (2014 *REStud*) reported powerful experimental evidence for this level- $k$  model from effectively zero-sum three-state betting games.
- Crawford and Iriberri (2007 *ECMA*) showed that this level- $k$  model describes most subjects' overbidding and vulnerability to the winner's curse in initial responses in auction experiments
- Brown, Camerer, and Lovo (2012 *AEJ Micro*) use this level- $k$  model to explain film-goers' failure to draw negative inferences from studios' withholding weak movies from critics pre-release.

## Level- $k$ analysis of the double auction

I first apply the level- $k$  model to CS's trading environment, focusing on their leading example with uniform value densities.

I assume that a trader's level is independent of its value.

I set  $L0$ 's frequency to zero, and focus on homogeneous populations of  $L1$ s or  $L2$ s, which allows simple illustrations of the main points.

Denote the buyer's bidding strategy  $b_i(V)$  and the seller's asking strategy  $a_i(C)$ , where the subscripts denote levels  $i = 1, 2$ .

## ***L1s' optimism, aggressiveness, and incentive-inefficiency***

An *L1* buyer believes that the seller's ask is uniformly distributed on  $[0, 1]$ , independent of its value.

Optimization then yields  $b_1(V) = 2V/3$  (in the interior).

Similarly, an *L1* seller's ask  $a_1(C) = 2C/3 + 1/3$  (in the interior).

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(Compare Crawford and Iriberri's 2007 *ECMA* analysis of *L1* bidding in first-price auctions.)

Despite the multiplicity of equilibria in the double auction, a level- $k$  model makes generically unique predictions, conditional on the population level frequencies.

*L1*'s optimal strategy is independent of the value densities; unlike *L2*'s, which depends on the other trader's density, or an equilibrium strategy, which depends on both traders' densities.

With  $b_1(V) = 2V/3$ , an *L1* buyer bids  $1/12$  more aggressively (that is, bids less) than an equilibrium buyer with  $b_*(V) = 2V/3 + 1/12$ .

With  $a_1(C) = 2C/3 + 1/3$ , an *L1* seller asks  $1/12$  more aggressively (more) than an equilibrium seller with  $a_*(C) = 2C/3 + 1/4$ .

Both have the same  $2/3$  shading as equilibrium bids or asks.

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For an *L1* buyer and seller, trade takes place iff  $V \geq C + 1/2$ ; ex post efficiency is lost for more values than in equilibrium with  $V \geq C + 1/4$ ; and the ex ante probability of trade is  $1/8 = 12.5\%$ , versus the equilibrium  $9/32 \approx 28\%$ .

Is there a mechanism that enhances efficiency for *L1*s by counteracting their aggressiveness in the double auction?

I will show that, whether or not *L1*-incentive-compatibility is required, the answer is Yes.

## ***L2s' pessimism, unaggressiveness, incentive-superefficiency***

An *L2* buyer's bid  $b_2(V)$  maximizes over  $b \in [0, 1]$

$$\int_0^b \left[ V - \frac{a + b}{2} \right] g(a_1^{-1}(a)) da + \int_b^1 0 da,$$

where  $g(a_1^{-1}(a))$  is the density of an *L1* seller's ask  $a_1(C)$  induced by the value density  $g(C)$ .

For instance, if  $g(C)$  is uniform, an *L2* buyer believes that the seller's ask  $a_1(C) = 2C/3 + 1/3$  is uniformly distributed on  $[1/3, 1]$ , with density  $3/2$  there and 0 elsewhere.



An  $L2$  buyer who believes the seller's ask is distributed on  $[1/3, 1]$  believes that trade requires  $b > 1/3$ .

For  $V \leq 1/3$  it is then optimal for an  $L2$  buyer to bid anything it thinks yields 0 probability of trade: In the absence of dominance among such strategies, I set  $b_2(V) = V$  for  $V$  in  $[0, 1/3]$ .

For  $V > 1/3$ , an  $L2$  buyer's bid  $b_2(V)$  maximizes over  $b \in [1/3, 1]$

$$\int_{1/3}^b \left[ V - \frac{a+b}{2} \right] (3/2) da.$$

The second-order condition is always satisfied.

Solving the first-order condition  $(3/2)(V - b) - (3/4)(V - 1/3) = 0$  yields  $b_2(V) = 2V/3 + 1/9$  for  $V \in [1/3, 1]$ .

Similarly, an  $L2$  seller's ask  $a_2(C) = 2C/3 + 2/9$  (in the interior).

With  $b_2(V) = 2V/3 + 1/9$ , an  $L2$  buyer bids  $1/36$  less aggressively (more) than an equilibrium buyer with  $b_*(V) = 2V/3 + 1/12$  and  $1/9$  less aggressively (more) than an  $L1$  buyer with  $b_1(V) = 2V/3$ .

With  $a_2(C) = 2C/3 + 2/9$ , an  $L2$  seller asks  $1/36$  less aggressively (less) than an equilibrium seller, and  $1/9$  less than an  $L1$  seller.

Both again have the same  $2/3$  shading as equilibrium or  $L1$ .

For an  $L2$  buyer and  $L2$  seller, trade takes place iff  $V \geq C + 1/6$ ; ex post efficiency is lost for fewer values than in equilibrium with  $V \geq C + 1/4$  or with  $L1$ s with  $V \geq C + 1/2$ ; and the ex ante probability of trade is  $25/72 \approx 35\%$ , versus the equilibrium  $28\%$  or  $L1$   $12.5\%$ .

Is there a mechanism that does as well or better for  $L2$ s than the double auction by further exploiting their *unaggressiveness*?

I will show that, whether or not  $L2$ -incentive-compatibility is required, the answer is No, at least for uniform value densities.

## **Design requiring level- $k$ -incentive-compatibility (not wlog)**

Throughout the analysis of level- $k$  design, I restrict attention to direct mechanisms, whose decisions can be viewed as estimates of own values; and I ignore the noisiness of people's decisions.

I define incentive-efficiency notions for a designer's correct beliefs; but I derive incentive constraints from traders' level- $k$  beliefs.

I use "incentive-compatible" here in the narrow sense, for direct mechanisms in which it is optimal for people to report truthfully.

But when I relax it traders are still assumed to best respond, even if such responses need not be truthful, as in a first-price auction.

"Level- $k$ -incentive-compatibility" and "level- $k$ -interim-individual-rationality" parallel the standard notions, "equilibrium-incentive-compatibility" and "equilibrium-interim-individual-rationality".

## **Level- $k$ -incentive-compatibility with levels observable**

Recall that MS's Theorems 1 and 2 use conditions for equilibrium-incentive-compatibility to derive an incentive budget constraint, subject to which an equilibrium-incentive-efficient mechanism must (for traders with quasilinear utility functions) maximize the sum of traders' ex ante expected utilities.

MS's Corollary 1 shows that no incentive-compatible individually rational mechanism can assure ex post Pareto-efficiency with probability one.

In CS's example with uniform value densities, MS's Theorems 1-2 yield a closed-form solution for the incentive-compatible form of their incentive-efficient mechanism, which transfers the object when the reported values satisfy  $v \geq c + \frac{1}{4}$ , at price  $(v + c + \frac{1}{2})/3$ .

When  $Lk$ -incentive-compatibility is required, MS's characterization of the incentive-efficient mechanism with uniform value densities is completely robust to level- $k$  thinking:

*Theorem 1. With uniform value densities, MS's equilibrium-incentive-efficient direct mechanism is efficient in the set of level- $k$ -incentive-compatible mechanisms for any population of level- $k$  traders with  $k > 0$ , with levels observable or not.*

Proof. Induction from the facts that, with uniform value densities,  $L1$  traders have the same beliefs as in the truthful equilibrium of MS's equilibrium-incentive-efficient mechanism; and both objective functions are based on the same, correct beliefs.

## **“Level- $k$ menu effects” and failures of the revelation principle**

MS showed that with uniform value densities, their incentive-efficient mechanism yields the same outcomes as CS's linear double-auction equilibrium, with traders shading their bids and asks to mimic the effect of truthful reporting in MS's mechanism.

This illustrates the revelation principle, whereby equilibrium makes the choice between those mechanisms neutral.

But my examples with uniform value densities show that the choice is not neutral for level- $k$  traders:

In the double auction  $L1$ s do worse than in MS's mechanism, while  $L2$ s do better.

Even for direct mechanisms and with  $L0$  fixed, the choice of mechanism influences the correctness of level- $k$  beliefs, via Crawford et al.'s (2009 *JEEA*) “level- $k$  menu effects”:

- MS's incentive-compatible, incentive-efficient mechanism neutralizes  $L1$ s' aggressiveness in the double auction by rectifying their beliefs.
- The non-incentive-compatible double auction (if feasible) improves upon MS's incentive-efficient mechanism for  $L2$ s by not rectifying their beliefs, preserving their *unaggressiveness*.

Level- $k$  menu effects are residues of  $Lk$ 's anchoring its beliefs on  $L0$ , whose influence is not eliminated by equilibrium thinking.

As a result, requiring level- $k$ -incentive-compatibility is not wlog.

Some analysts of design, e.g. in school choice or combinatorial auctions, have argued that incentive-compatibility is essential in applications, but mostly in equilibrium analyses where it is wlog.

Other analysts are willing to consider non-incentive-compatible mechanisms like the Boston Mechanism or first-price auctions.

I don't try to resolve this empirical question here.

Instead I first require  $Lk$ -incentive-compatibility.

I then consider relaxing it to allow any direct mechanism, while maintaining the assumption that traders best respond.



## Level- $k$ -incentive-compatibility with levels observable

With general value densities, the payoff-relevant aspects of a direct mechanism are still outcome functions  $p(\cdot, \cdot)$  and  $x(\cdot, \cdot)$ , where buyer and seller report values  $v$  and  $c$ , and  $p(v, c)$  is the probability the object transfers, for expected payment  $x(v, c)$ .

For a mechanism  $(p, x)$ ,  $f^k(v; p, x)$  and  $F^k(v; p, x)$  are the density and distribution function of an  $Lk$  seller's beliefs and  $g^k(c; p, x)$  and  $G^k(c; p, x)$  of an  $Lk$  buyer's.

With  $L0$  uniform on  $[0, 1]$ ,  $f^1(v; p, x) \equiv 1$  and  $g^1(c; p, x) \equiv 1$ .

If  $\beta_1(V; p, x)$  is an  $L1$  buyer's response to  $(p, x)$  with value  $V$  and  $\alpha_1(C; p, x)$  is an  $L1$  seller's response to  $(p, x)$  with cost  $C$ ,  
 $f^2(v; p, x) \equiv f(\beta_1^{-1}(v; p, x))$  and  $g^2(c; p, x) \equiv g(\alpha_1^{-1}(c; p, x))$ .

For ease of notation only, assume that each trader population is concentrated on one level.

As in MS's analysis, we can write the buyer's and seller's expected monetary payments, probabilities of trade, and utilities as functions of their value reports  $v$  and  $c$ .

$$\begin{aligned}
 X_B^k(v) &= \int_0^1 x(v, \hat{c}) g^k(\hat{c}) d\hat{c}, & X_S^k(c) &= \int_0^1 x(\hat{v}, c) f^k(\hat{v}) d\hat{v}, \\
 (5.1) \quad P_B^k(v) &= \int_0^1 p(v, \hat{c}) g^k(\hat{c}) d\hat{c}, & P_S^k(c) &= \int_0^1 p(\hat{v}, c) f^k(\hat{v}) d\hat{v}, \\
 U_B^k(V, v) &= VP_B^k(v) - X_B^k(v), & U_S^k(C, c) &= X_S^k(c) - CP_S^k(c).
 \end{aligned}$$

For a given  $k$ , the mechanism  $p(\cdot, \cdot)$ ,  $x(\cdot, \cdot)$  is  $Lk$ -incentive-compatible iff truthful reporting is optimal given  $Lk$  beliefs.

That is, if for every  $V$ ,  $v$ ,  $C$ , and  $c$  in  $[0, 1]$ ,

$$(5.2) \quad U_B^k(V, V) \geq U_B^k(V, v) = VP_B^k(v) - X_B^k(v) \text{ and} \\ U_S^k(C, C) \geq U_S^k(C, c) = X_S^k(c) - CP_S^k(c).$$

For a given  $k$ , the mechanism  $p(\cdot, \cdot), x(\cdot, \cdot)$  is  $Lk$ -incentive-compatible iff truthful reporting is optimal given  $Lk$  beliefs.

That is, if for every  $V, v, C$ , and  $c$  in  $[0, 1]$ ,

$$(5.2) \quad U_B^k(V, V) \geq U_B^k(V, v) = VP_B^k(v) - X_B^k(v) \text{ and} \\ U_S^k(C, C) \geq U_S^k(C, c) = X_S^k(c) - CP_S^k(c).$$

The mechanism  $p(\cdot, \cdot), x(\cdot, \cdot)$  is (interim)  $Lk$ -individually rational iff for every  $V$  and  $C$  in  $[0, 1]$ ,

$$(5.3) \quad U_B^k(V, V) \geq 0 \text{ and } U_S^k(C, C) \geq 0.$$

Theorems 2 and 3 extend MS's (Theorems 1-2) characterization of equilibrium-incentive-efficient mechanisms to level- $k$  models in which the designer can observe traders' levels, so s/he can enforce a different mechanism for each pair of levels  $i$  and  $j$ .

**Theorem 2.** *Assume that the designer knows or can observe individual traders' levels  $i$  and  $j$ . Then, for any mechanism that is incentive-compatible for traders of those levels,*

$$(5.4) \quad U_B^i(0,0) + U_S^j(1,1) = \min_{V \in [0,1]} U_B^i(V,V) + \min_{C \in [0,1]} U_S^j(C,C) \\ = \int_0^1 \int_0^1 \left( \left[ V - \frac{1 - F^j(V)}{f^j(V)} \right] - \left[ C + \frac{G^i(C)}{g^i(C)} \right] \right) p(V,C) g^i(C) f^j(V) dC dV.$$

*And if  $p(\cdot, \cdot)$  is any function mapping  $[0, 1] \times [0, 1]$  into  $[0, 1]$ , there exists a function  $x(\cdot, \cdot)$  such that  $(p, x)$  is incentive-compatible and interim individually rational iff  $P_B^i(\cdot)$  is weakly increasing for all  $(p, x)$ ,  $P_S^j(\cdot)$  is weakly decreasing for all  $(p, x)$ , and*

$$(5.5) \quad 0 \leq \int_0^1 \int_0^1 \left\{ \left[ V - \frac{1 - F^j(V)}{f^j(V)} \right] - \left[ C + \frac{G^i(C)}{g^i(C)} \right] \right\} p(V,C) g^i(C) f^j(V) dC dV.$$

**Proof.** The proof follows MS's, adjusted for nonequilibrium beliefs. By (5.1),  $P_B^i(\cdot)$  is weakly increasing and  $P_S^j(\cdot)$  is weakly decreasing for any given  $(p, x)$ , which yields necessary and sufficient conditions for incentive-compatibility:

$$(5.6) \quad U_B^i(V, V) = U_B^i(0,0) + \int_0^V P_B^i(\hat{v})d\hat{v} \text{ for all } V \text{ and}$$

$$U_S^j(C, C) = U_S^j(1,1) + \int_C^1 P_S^j(\hat{c})d\hat{c} \text{ for all } C$$

(5.6) implies that  $U_B^i(V, V)$  is weakly increasing and  $U_S^j(C, C)$  is weakly decreasing, and shows that  $U_B^i(0,0) \geq 0$  and  $U_S^j(1,1) \geq 0$  suffice for individual rationality for all  $V$  and  $C$  as in (5.3).

Given (5.3) and that  $P_B^j(\cdot)$  is weakly increasing and  $P_S^k(\cdot)$  is weakly decreasing, arguments like MS's show that the transfer function,

$$(5.8) x(v, c) = \int_0^V v d[P_B^i(v)] - \int_0^C c d[-P_S^j(c)] + \int_0^1 c[1 - G^i(C)] d[-P_S^j(c)],$$

makes  $(p, x)$  incentive-compatible and interim individually rational for traders' levels. Q.E.D.

The analogue of MS's Corollary 1 goes through for level- $k$  models, with or without the assumption that the designer knows traders' levels.

**Corollary 1.** *Assume traders' levels are homogeneous or heterogeneous and observable or unobservable by the designer. If buyer's and seller's values are distributed with positive probability density over  $[0, 1]$ , then no level- $k$ -incentive-compatible and level- $k$ -interim individually rational trading mechanism can assure ex post efficiency with probability 1.*

**Proof.** The proof adapts MS's proof. First assume the designer knows the buyer is level  $i$  and the seller is level  $j$ . With positive value densities on  $[0, 1]$ , when  $p(V, C) \equiv 1$  iff  $V \geq C$ , the constraint (5.5) reduces to:

$$\begin{aligned}
 (5.9) \quad 0 &\leq \int_0^1 \int_0^1 \left\{ \left[ V - \frac{1-F^j(V)}{f^j(V)} \right] - \left[ C + \frac{G^i(C)}{g^i(C)} \right] \right\} p(V, C) g^i(C) f^j(V) dC dV = \\
 &\int_0^1 \int_0^V [V f^j(V) - 1 + F^j(V)] g^i(C) dC dV - \int_0^1 \int_0^V [C g^i(C) + G^i(C)] dC f^j(V) dV \\
 &= \int_0^1 [V f^j(V) - 1 + F^j(V)] G^j(V) dV - \int_0^1 [V G^i(V) f^j(V)] dV \\
 &= - \int_0^1 [1 - F^j(V)] G^i(V) dV < 0, \text{ a contradiction.}
 \end{aligned}$$

Unobservable levels add screening constraints, so cannot help. Q.E.D.

Theorem 3 gives a concrete characterization of mechanisms that are efficient in the set of level- $k$ -incentive-compatible mechanisms when buyer's and seller's levels are observable by the designer.

Define, for  $\beta \geq 0$ ,

$$\begin{aligned} \psi^{ij}(V, C; \beta) &= \left[ V - \beta \frac{1 - F^j(V)}{f^j(V)} \right] \left[ \frac{g^i(C)f^j(V)}{g(C)f(V)} \right] - \left[ C + \beta \frac{G^i(C)}{g^i(C)} \right] \left[ \frac{g^i(C)f^j(V)}{g(C)f(V)} \right] \\ (5.10) \quad &= (V - C) - \beta \left[ \frac{g^i(C)f^j(V)}{g(C)f(V)} \right] \left[ \frac{1 - F^j(V)}{f^j(V)} + \frac{G^i(C)}{g^i(C)} \right], \text{ and} \\ p_\beta^{ij}(V, C) &= 1 \text{ if } \psi^{ij}(V, C; \beta) \geq 0, \text{ and } p_\beta^{ij}(V, C) = 0 \text{ if } \psi^{ij}(V, C; \beta) \leq 0. \end{aligned}$$

If feasible,  $p_0^{ij}(V, C)$  would yield an ex post efficient allocation; but Corollary 1 shows that it violates the incentive budget constraint (5.5).  $p_1^{ij}(V, C)$  maximizes the slack in (5.5), but wastes surplus.

The goal is an optimal compromise between these two extremes.



**Theorem 3.** *Assume that the designer knows or can observe individual traders' levels,  $i$  for the buyer and  $j$  for the seller. If there exists a level- $k$ -incentive-compatible mechanism  $(p, x)$  such that  $U_B^i(0,0) = U_S^j(1,1) = 0$  and  $p = p_\beta^{ij}(V, C)$  for some  $\beta \in [0, 1]$ , then that mechanism maximizes traders' true ex ante expected total gains from trade among all level- $k$ -incentive-compatible and level- $k$ -interim individually rational mechanisms. Furthermore, if  $\Psi^{ij}(V, C; 1)$  is increasing in  $V$  and decreasing in  $C$  for any given  $(p, x)$ , then such a mechanism must exist.*

**Proof.** The proof adapts MS's proof. Fix buyer's and seller's levels  $i$  and  $j$ , and consider choosing  $p(\cdot, \cdot)$  to maximize traders' ex ante expected total gains from trade subject to  $0 \leq p(\cdot, \cdot) \leq 1$ ,  $U_B^i(0,0) = U_S^j(1,1) = 0$ , and (5.5). (5.5) and (5.10) yield:

$$(5.11) \quad \max_{\{0 \leq p(\cdot, \cdot) \leq 1\}} \int_0^1 \int_0^1 (V - C) p(V, C) g(C) f(V) dC dV$$

$$\text{s.t. } 0 \leq \int_0^1 \int_0^1 \Psi^{ij}(V, C; 1) p(V, C) g(C) f(V) dC dV.$$

If a solution  $p(\cdot, \cdot)$  to (5.11) yields a  $P_B^i(\cdot)$  that is weakly increasing for all  $v$  and a  $P_S^j(\cdot)$  that is weakly decreasing for all  $c$ , then by Theorem 2 that solution is associated with a mechanism that maximizes traders' ex ante expected total gains from trade among all level- $k$ -incentive-compatible and level- $k$ -interim individually rational mechanisms.

(5.11) is like a consumer's budget problem, with trade probabilities  $p(V, C)$  like a continuum of goods and "prices"  $\left\{ \left[ V - \frac{1-F^j(V)}{f^j(V)} \right] - \left[ C + \frac{G^i(C)}{g^i(C)} \right] \right\}$ . Because the  $p(V, C)$  enter the objective function and the constraint linearly, there are solutions that are "bang-bang", with  $p(V, C) = 0$  or  $1$  almost everywhere and  $p(V, C) = 1$  for the  $(V, C)$  pairs with the largest expected gain per unit of incentive cost (as for the highest marginal-utility-to-price ratios).

By Corollary 1, with  $V$  and  $C$  continuously distributed, the constraint holds with equality at a solution.

Form the Lagrangean:

$$\begin{aligned}
 & \int_0^1 \int_0^1 (V - C) p(V, C) g(C) f(V) dC dV \\
 & + \lambda \int_0^1 \int_0^1 \Psi^{ij}(V, C; 1) p(V, C) g(C) f(V) dC dV \\
 (5.12) \quad & = \int_0^1 \int_0^1 \left( V - C + \lambda \Psi^{ij}(V, C; 1) \right) p(V, C) g(C) f(V) dC dV \\
 & = (1 + \lambda) \int_0^1 \int_0^1 \left( \Psi^{ij} \left( V, C; \frac{\lambda}{1 + \lambda} \right) p(V, C) g(C) f(V) dC dV \right).
 \end{aligned}$$

Any function  $p(V, C)$  and  $\lambda \geq 0$  that satisfy the constraint with equality and the Kuhn-Tucker conditions solves problem (5.11).

The Kuhn-Tucker conditions are:

$$(5.13) \quad (1 + \lambda)\Psi^{ij} \left( V, C; \frac{\lambda}{1+\lambda} \right) \leq 0 \text{ or equivalently } (V - C) - \frac{\lambda}{1+\lambda} \left[ \frac{g^i(C)f^j(V)}{g(C)f(V)} \right] \left[ \frac{1-F^j(V)}{f^j(V)} + \frac{G^i(C)}{g^i(C)} \right] \leq 0, \text{ when } p(V, C) = 0, \text{ and}$$

$$(5.14) \quad (1 + \lambda)\Psi^{ij} \left( V, C; \frac{\lambda}{1+\lambda} \right) \geq 0 \text{ or equivalently } (V - C) - \frac{\lambda}{1+\lambda} \left[ \frac{g^i(C)f^j(V)}{g(C)f(V)} \right] \left[ \frac{1-F^j(V)}{f^j(V)} + \frac{G^i(C)}{g^i(C)} \right] \geq 0, \text{ when } p(V, C) = 1.$$

Given the continuity and monotonicity of  $\Psi^{ij}(V, C; \beta)$ , there is a unique  $\lambda$  and  $p = p_{\beta}^{ij}(V, C)$ , with  $\beta = \frac{\lambda}{1+\lambda}$  (equivalently,  $\lambda = \frac{\beta}{1-\beta}$ ), that satisfy  $U_B^i(0,0) = U_S^j(1,1) = 0$ , (5.5), (5.13), and (5.14). Q.E.D.

Theorem 3's condition that  $\Psi^{ij}(V, C; 1)$  is increasing in  $V$  and decreasing in  $C$  for all  $(p, x)$  is the level- $k$  analogue of MS's Theorem 2 monotonicity conditions, which is satisfied whenever the true densities fit Myerson's (1981) "regular case", ruling out strong hazard rate variations in the "wrong" direction.

If traders' beliefs  $f^j(V; p, x)$  and  $g^i(C; p, x)$  were equal to the true densities  $f(V)$  and  $g(C)$ , Theorem 3's condition reduces to MS's Theorem 2 condition.

Theorem 3's condition jointly restricts the beliefs and the true densities in a similar but distinct way.

The proofs of Theorems 2 and 3 and Corollary 1 show that analogous results go through for any behavioral model that makes unique predictions that are best responses to beliefs and satisfies analogous monotonicity restrictions.

## **Properties of mechanisms that are efficient in the set of $Lk$ -incentive-compatible mechanisms**

Comparing level- $k$  and equilibrium incentive budget constraints ((5.5) with (2.5) (MS's (2))), and the level- $k$  and equilibrium Kuhn-Tucker conditions ((5.14) with (2.8) (MS's p. 274)) shows that the design features that foster equilibrium-incentive-efficiency foster efficiency in the set of level- $k$ -incentive-compatible mechanisms, with different weights due to incentive effects of different beliefs.

The level- $k$  Kuhn-Tucker conditions (5.14) reveal another design feature that fosters efficiency in the set of level- $k$ -incentive-compatible mechanisms:

Unless such a mechanism happens to induce correct beliefs (as it does with uniform value densities, by Theorem 1), it must use tacit exploitation of predictably incorrect beliefs (“TEPIB”):

- “Predictably” via the level- $k$  model.
- “Exploitation” in the benign sense that traders’ incorrect beliefs are used only for their benefit.
- “Tacit” in that the mechanism does not actively deceive traders.

Suppose one could exogenously increase the pessimism of traders' level- $k$  beliefs relative to the truth, e.g. lower bids or higher asks, in the sense of first-order stochastic dominance.

That would loosen the incentive budget constraint (5.5), and increase maximized expected total surplus.

It is not possible to change traders' beliefs exogenously, but the tradeoffs in (5.5) reflect the influence of pessimism or optimism.

Relative to an equilibrium-incentive-efficient mechanism, TEPIB favors trade at  $(V, C)$  combinations for which traders' non-equilibrium beliefs make the "prices"  $\left[ \frac{g^i(C)f^j(V)}{g(C)f(V)} \right] \left[ \frac{1-F^j(V)}{f^j(V)} + \frac{G^i(C)}{g^i(C)} \right]$  low compared to those in MS's equilibrium analysis  $\left[ \frac{1-F(V)}{f(V)} + \frac{G(C)}{g(C)} \right]$ .

For  $L2$  and higher levels, TEPIB also favors mechanisms that increase the advantages of the first two effects.



TEPIB also suggests that viewing “robust” mechanism design as achieving equilibrium-incentive-efficient outcomes under weaker behavioral assumptions may be too narrow.

For example:

- A second-price auction seems more robust than an equilibrium-revenue-equivalent first-price auction, because it yields the equilibrium outcome for any mixture of level- $k$  bidders.
- But auction design for  $L1$ s (at least) tends to favor first-price auctions, which make  $L1$ s overbid, yielding revenue higher than in equilibrium or in a second-price auction, which makes  $L1$  bidders mimic equilibrium (Crawford and Iriberri 2007, CKNP).

## Examples of mechanisms that are efficient in the set of level- $k$ -incentive-compatible mechanisms

As in MS's analysis, closed-form solutions are available only with uniform value densities, for which TEPIB has no influence.

To illustrate TEPIB, I report computed trading regions for such mechanisms for  $L1$ s and combinations of linear densities.

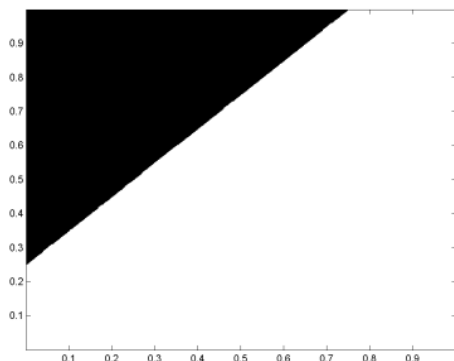
(The paper's Figure 1 reports  $L1$ s' trading regions for a coarse subset of linear density combinations, excluding only extreme combinations that violate Theorems 2-3's monotonicity conditions.

For  $L2$ s, with  $f^2(v) \equiv f(\beta_1^{-1}(v; p, x))$  and  $g^2(c) \equiv g(\alpha_1^{-1}(c; p, x))$ , (5.5) and (5.14) depend on both the transfer function  $x(\cdot, \cdot)$  and  $p(\cdot, \cdot)$ , making the dimensionality of search too high.)

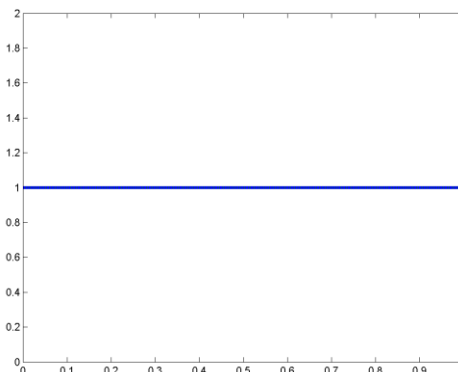
In this case mechanisms that are efficient in the set of  $L1$ -incentive-compatible mechanisms are similar to equilibrium-incentive-efficient mechanisms in most respects.

**From Figure 1. Trading regions (in black) for equilibrium-incentive-efficient mechanisms and mechanisms that are efficient in the set of  $L1$ -incentive-compatible mechanisms**  
 (Buyer's value  $V$  is on the vertical axis; seller's value  $C$  is on the horizontal axis. All value densities are linear; " $x, y$ " means the buyer's density  $f(V)$  satisfies  $f(0) = x$  and  $f(1) = 2-x$ , and the seller's density  $g(C)$  satisfies  $g(0) = y$  and  $g(1) = 2-y$ .)

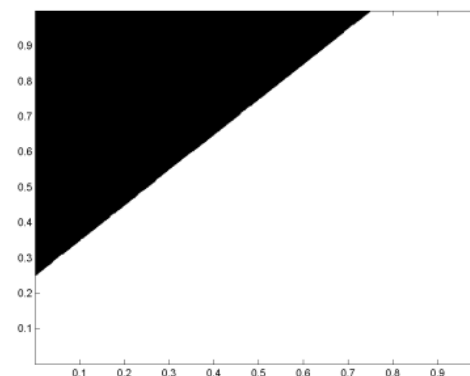
**Equilibrium: 1.0, 1.0**



**Buyer (—) and seller (⋯)**

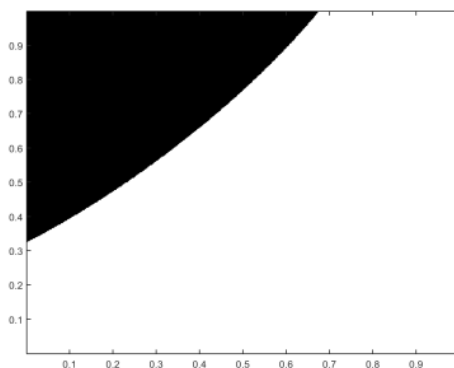


**L1: 1.0, 1.0**

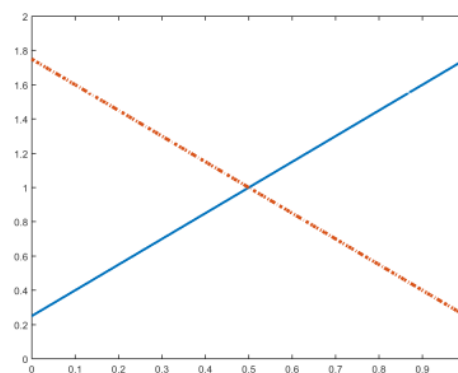


**Uniform value densities**

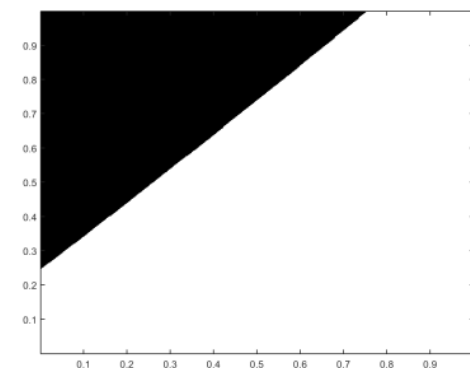
**Equilibrium: 0.25, 1.75**



**Buyer (—) and seller (⋯)**

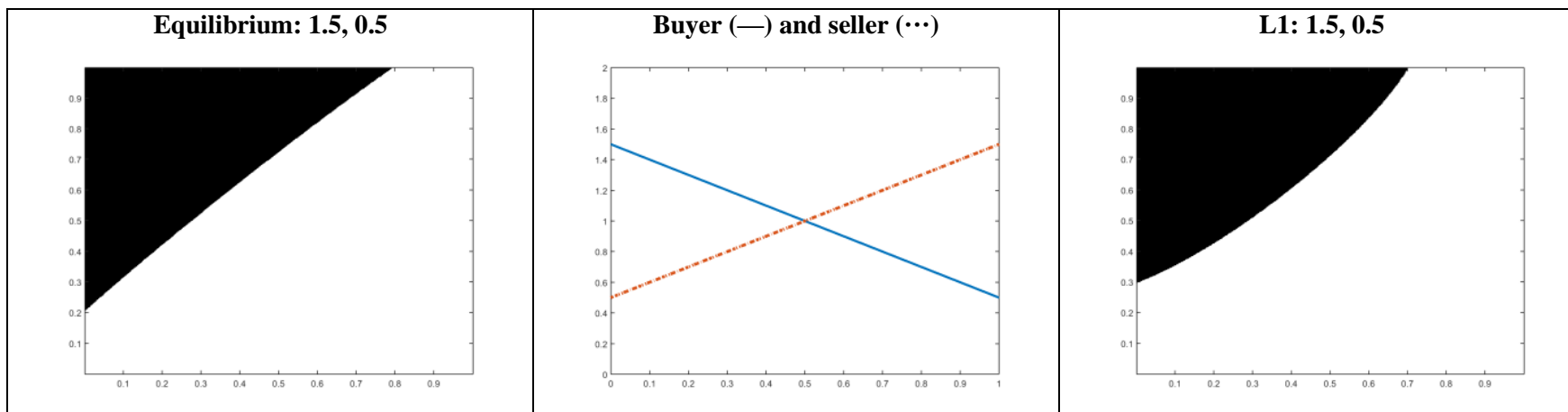


**L1: 0.25, 1.75**

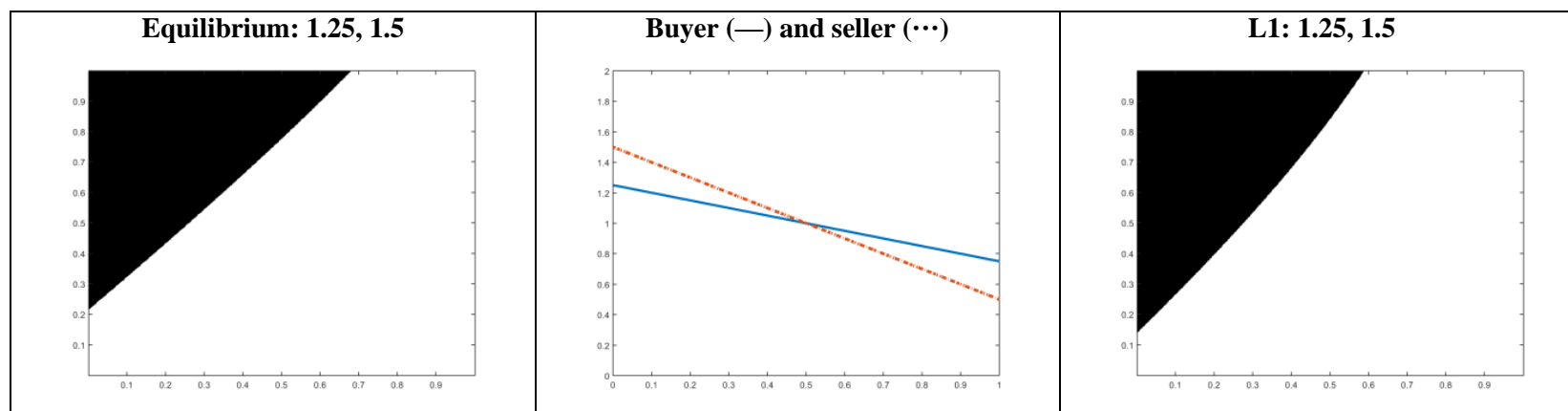


**Pessimism makes  $L1$  trading region larger than for equilibrium**

**From Figure 1. Trading regions (in black) for equilibrium-incentive-efficient mechanisms and mechanisms that are efficient in the set of  $L1$ -incentive-compatible mechanisms**  
 (Buyer's value  $V$  is on the vertical axis; seller's value  $C$  is on the horizontal axis. All value densities are linear; " $x, y$ " means the buyer's density  $f(V)$  satisfies  $f(0) = x$  and  $f(1) = 2-x$ , and the seller's density  $g(C)$  satisfies  $g(0) = y$  and  $g(1) = 2-y$ .)



**Optimism makes  $L1$  trading region smaller than for equilibrium**



**Optimism/pessimism intuition fails: overlapping trading regions**

## **Level- $k$ -incentive-compatibility with levels unknown and unobservable, and heterogeneous**

I now continue to assume that level- $k$ -incentive-compatibility is required, but relax the assumption that individual traders' levels are known or observable and homogeneous.

Theorem 1 shows that with uniform value densities, even in this case, MS's equilibrium-incentive-efficient direct mechanism is efficient in the set of level- $k$ -incentive-compatible mechanisms.

Corollary 1 shows that with general value densities that are positive over  $[0, 1]$ , even in this case, no level- $k$ -incentive-compatible and level- $k$ -interim individually rational trading mechanism can assure ex post efficiency with probability 1.

A “random posted-price mechanism” is a distribution over prices  $\pi$  and a function  $\mu(\cdot)$  such that trade occurs at price  $\pi$  with probability  $\mu(\pi)$  iff  $v \geq \pi \geq c$ , with no trade or transfer otherwise.

**Theorem 4.** *Assume that the designer knows that the population level distributions for buyers and sellers include, with positive probabilities, L1 and at least one higher level; but s/he cannot observe individual traders’ levels. Then, if  $f(\cdot)$ ,  $g(\cdot) \neq 1$  almost everywhere, level-k-incentive-compatibility requires that the mechanism is equivalent to a random posted-price mechanism. Further, a mechanism that maximizes traders’ true expected total gains from trade among level-k-incentive-compatible and interim individually rational mechanisms is equivalent to a deterministic posted-price mechanism with  $U_B^i(0,0) = U_S^j(1,1) = 0$  for all levels  $i$  in the buyer and  $j$  in the seller population. The optimal posted price  $\pi$  is unique, characterized by the first-order condition:*

$$(5.15) \quad \frac{f(\pi)}{g(\pi)} = \frac{\int_{\pi}^1 (V-\pi)f(V)dV}{\int_0^{\pi} (\pi-C)g(C)dc} = \frac{E(V-\pi|V \geq \pi)}{E(\pi-C|C \leq \pi)}.$$

**Proof.** By Theorem 2, (5.6) must hold for any levels in the buyer and seller populations. If a mechanism is  $L1$ -incentive-compatible, the proof of Theorem 1 shows that it is  $Lk$ -incentive-compatible for all  $k > 1$  if and only if it is also equilibrium-incentive-compatible:

$$(5.6) \quad U_B^i(V, V) = U_B^i(0,0) + \int_0^V P_B^i(\hat{v})d\hat{v} \text{ for all } V \text{ and}$$

$$U_S^j(C, C) = U_S^j(1,1) + \int_C^1 P_S^j(\hat{c})d\hat{c} \text{ for all } C$$

Standard arguments show that that is a contradiction unless the mechanism is equivalent to a random posted-price mechanism. A random posted-price mechanism is level- $k$ -incentive-compatible for any  $k$ . Linearity implies that there are always “bang-bang”  $p(v, c)$  solutions with  $p(V, C) = 0$  or  $1$  almost everywhere, so  $\mu(\cdot) \equiv 1$  wlog. An optimal deterministic posted-price mechanism solves:

$$(5.25) \quad \max_{\{0 \leq \pi \leq 1\}} \int_{\pi}^1 \int_0^{\pi} (V - C) p(V, C) g(C) f(V) dC dV.$$

The second-order condition is satisfied globally, so there is a uniquely optimal posted price, characterized by the first-order condition (5.15). Q.E.D.

Theorem 4 shows that the need to screen traders' levels as well as their values rules out the sensitive dependence on reported values of mechanisms that are equilibrium-incentive-efficient or efficient in the set of level- $k$ -incentive-compatible mechanisms when traders' levels are known or observable: a plausible, basic rationale for distribution-free, dominant-strategy implementation often assumed in the literature on robust mechanism design.

Theorem 4's result continues to hold for any sufficiently well-behaved nonequilibrium model where strategic thinking falls into identifiable classes, and which makes unique predictions that can be viewed as best responses to some beliefs.



The case of approximately uniform value densities gives an idea of the cost of giving up sensitive dependence on reported values.

The equilibrium-incentive-efficient mechanism then approximately yields ex ante probability of trade  $9/32 \approx 28\%$  and expected surplus  $9/64 \approx 0.14$ . The optimal posted price with uniform densities is  $1/2$ , which then yields probability of trade  $1/4 = 25\%$  and expected surplus  $1/8 = 0.125$ , a modest cost for robustness.

Such static posted-price mechanisms come closer to satisfying Wilson's (1987) desideratum, in that their *rules* are distribution-free. However, the optimal posted price is determined, via (5.15), by conditional means that depend on the full value densities.

Čopič and Ponsatí (2015) note that a dynamic implementation via continuous-time double auction with bids revealed to traders only once they are compatible can avoid such dependence, in that any (even random) posted-price mechanism can be so implemented.

## Relaxing level- $k$ -incentive-compatibility

In this case one can still define a general class of feasible direct mechanisms; and the payoff-relevant aspects of a mechanism are still described by outcome functions  $p(\cdot, \cdot)$  and  $x(\cdot, \cdot)$ .

However, even a direct mechanism's incentive effects can no longer be tractably captured via incentive constraints. Instead they must be modeled via level- $k$  traders' best responses to it.

## Known or observable levels

Here, for tractability, I focus on double auctions with reserve prices chosen by the designer, and on uniform value densities.

Reserve prices have no benefit when  $L_0$  is uniform random on the full range of possible values  $[0, 1]$ , as assumed so far.

But a restricted menu might make  $L_k$  players anchor beliefs instead on the correspondingly restricted range of bids or asks, which can make reserve prices useful (CKNP).

For example, in the double auction with uniform value densities,  $L1$  traders believe they face bids or asks uniformly distributed on  $[0, 1]$ , which leads to incentive-*inefficient* outcomes.

To implement the outcome of MS's equilibrium-incentive-efficient direct mechanism via the double auction,  $L1$  traders have to believe that they face bids or asks uniform on  $[1/4, 3/4]$ , the range of “serious” bids or asks in CS's linear double-auction equilibrium.

If  $L1$  traders anchor on the restricted menu, those beliefs can be induced by restricting bids to  $[1/4, 3/4]$  and asks to  $[1/4, 3/4]$ . (The upper ask limit could be moved to 1 and the lower bid limit to 0.)

Thus with uniform value densities, for  $L1$ s a double auction with reserve prices can mimic MS's equilibrium-incentive-efficient mechanism, whose direct form is then efficient in the set of  $L1$ -incentive-compatible mechanisms.

(MS's general specification of feasible mechanisms implicitly allows reserve prices, and their analysis therefore shows that if equilibrium is assumed, reserve prices are not useful here.)

For  $L2$ s with uniform value densities, my analysis of the double auction without reserve prices shows that it can improve upon a mechanism that is efficient in the set of  $L2$ -incentive-efficient mechanisms, or MS's equilibrium-incentive-efficient mechanism.

It can be shown that reserve prices allow no further improvement, so the double auction without reserve prices is optimal for  $L2$ s.

## **Unknown and unobservable, and heterogeneous levels**

Here suppose, again for tractability, that the designer knows that the population includes multiple levels with positive probability, but that the frequency of one, known level is very high.

Then a mechanism that would be efficient in the set of level- $k$ -incentive-compatible mechanisms for the frequent level, or perhaps a level- $k$ -incentive-efficient mechanism (relaxing level- $k$  incentive-compatibility) for that level, can generally improve upon a mechanism that is efficient in the set of level- $k$ -incentive-compatible mechanisms for the full heterogeneous population.

Such a mechanism gains the benefits of sensitive dependence on reported values for most traders, at bounded cost for the rest.

For example, in the case of approximately uniform value densities a mechanism that is efficient in the set of level- $k$ -incentive-compatible mechanisms is a posted-price mechanism, with approximate optimal posted price  $\frac{1}{2}$ , probability of trade  $\frac{1}{4} = 25\%$ , and expected total surplus  $\frac{1}{8} = 0.125$ .

By contrast, a mechanism that is efficient in the set of  $L1$ -incentive-compatible mechanisms with uniform value densities yields ex ante probability of trade  $\frac{9}{32} \approx 28\%$  and expected surplus  $\frac{9}{64} \approx 0.14$  for almost all traders, a significant gain.

Alternatively, the  $L2$ -incentive-efficient double auction without reserve prices yields ex ante probability of trade  $\frac{25}{72} \approx 35\%$  and expected surplus  $\frac{11}{72} \approx 0.15$ , an even larger gain.

More generally, relaxing the restriction to level- $k$ -incentive-compatible mechanisms can yield level- $k$ -incentive-efficient mechanisms that differ qualitatively as well as quantitatively from equilibrium-incentive-efficient mechanisms, with substantial gains in incentive-efficiency.